Peter Haïssinsky I

Setting: X cmpt metrizable space
$G < \mathrm{Homeo}(X)$

Def^n: Given $a, b \in X$, (g_n) is (a,b) collapsing if $g_n \to b$ uniformly on cmpt subsets of $X \setminus \{a\}$.

cvg of funct^n cvg to const function b.

Prop: TFAE:
1) Given any subset $\Phi \subset G$ either $\bar{\Phi}$ is cmpt or contains a collapsing seq
2) The diagonal action of G on the set of distinct triples in X is proper.
($\forall K, L$ of triples $\{g \in G \mid gK \cap L \neq \emptyset\}$ is cmpt

Def^n: G is a convergence group if one of these properties holds.

Rmk: If G is discrete then this means any ∞ seq contains a collapsing subsequence.

Notation: $\Theta(X) = \{(x_1, x_2, x_3) \in X^3,\ x_i \neq x_j\ i \neq j\}$

Proof: 1) $\Rightarrow$ 2) let K, L cmpt subsets of $\Theta(X)$
$A = \{g \in G \mid g(K) \cap L \neq \emptyset\}$.
If $\bar{A}$ not compact, then $\exists\ (g_n)$, $a, b \in X$ st
using ①.
$g_n|_{X - \{a\}} \longrightarrow b$

$\Theta(X)$ cmpt $\Rightarrow \exists\, \delta > 0$ st $\forall\ (x_j) \in L\ \ |x_i - x_j| > \delta$ } this prevents existence of collapsing seq in A.

2) $\Rightarrow$ 1) $\Phi \subset G$ st $\bar{\Phi}$ is not cmpt
i.e $\forall\ (x_j) \in \Theta(X)$, $\exists (g_n) \in \Phi$ and two $i, j \in \{1,2,3\}$ st
$g_n(x_i), g_n(x_j) \longrightarrow b \in X$

Comment: Notion comes from Gehring - Martin
Bowditch, Freden, Tukia

Example 1: Möbius transformations on $\hat{\mathbb{C}}$

$$M: \Theta(\hat{\mathbb{C}}) \longrightarrow \mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$$
$$(x_i) \longmapsto \left(z \longmapsto \frac{z - x_1}{z - x_2} \cdot \frac{x_1 - x_2}{x_3 - x_1} \right)$$
} homeo.

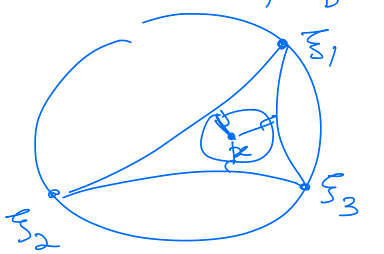
K, L cmpt $\subseteq \Theta(X)$ $\{g \in \mathrm{M\ddot{o}b}(\hat{\mathbb{C}}) \mid gK \cap L \neq \emptyset\} = \{h_2^{-1} h_1$ where h_1, h_2 are in $M(K) \cap M(L)\}$

cmpt set because M homeo.

Example 2: Z = proper geod hyp space
$G = \mathrm{Isom}(Z)$
∂Z visual ∂
$(\partial Z, d)$ is cmpt, $G \curvearrowright \partial Z$ by homeo

Claim: the action is a convergence action.

fix $D>0$, $G \curvearrowright Y_D := \{ (x, (\xi_j)) \in Z \times \Theta(\partial Z) \mid d(x, (\xi_i, \xi_j)) < \infty \}$



$x \in Z \curvearrowleft G$ $\qquad$ $(\xi_j) \in \Theta(\partial Z) \curvearrowleft G$

proper eq map

$G \curvearrowright Z$ proper $\Rightarrow$ $G \curvearrowright Y_D$ proper $\Rightarrow$ $G \curvearrowright \Theta(\partial Z)$ proper.

Example 3: Uniform quasi-Möbius groups

A quasi-Möbius map $f: X \to Y$ metric spaces
$\exists\, \eta: \mathbb{R}_+ \to \mathbb{R}_+$ distortion function (homeo)
st $\forall\, x_1, x_2, x_3, x_4 \in X$ distinct

$$\frac{|f(x_1)-f(x_2)|}{|f(x_1)-f(x_3)|}\,\frac{|f(x_4-\quad|}{|\quad-\quad|} \le \eta\left(\frac{|x_1-x_2|}{|x_1-x_4|}\,\frac{|\quad|}{|\quad|}\right)$$

Defn: G is unif qmorphism if $\exists\, \eta$ st $\forall g \in G$ is η-qMöbius.
Such groups are convergence groups.

Ex. Z hyp, $G = \mathrm{Isom}(Z)$ $\quad G \curvearrowright (\partial Z, d)$ is q-Möbius.
(visual metric)

Q. Isom(hyp space) $\Rightarrow$ unif q Möbius $\Rightarrow$ cvg group actn.
$\Leftarrow$? $\qquad$ $\Leftarrow$? find q metric

Thm (J. Ferrand ≈70) M closed Riemannian mfld
$\mathrm{Conf}(M) = \{$conformal diffeo of $M\}$ is a cvg group act.

Ex. G f.g. $\quad S$ = finite gen set
$Z = \mathrm{Cay}(G, S)$

fix $f:\mathbb{N}\to\mathbb{R}_+$ st $\exists\, \lambda\in(0,1)$

1) $\lambda f(n) \le f(n+1) \le f(n)$
2) $\sum f(n) < \infty$

$e=(x,y)$ $\quad d_f(e) = f\left(\min\{d(1,x), d(1,y)\}\right)$

$d_f(x,y) = \inf \ell_f$(curves joining x & y).

(Z, d_f) take the completion

$\bar{Z} \setminus Z = \partial_f Z$ Floyd ∂ of G

(Karlsson)
Prop: $G \curvearrowright \partial_f Z$ is a convergent action. (often $\bar{Z}$ = one-point compactificat^n).

Q For which group G is $\partial_f Z$ non-trivial?

Gerasimov: True for rel hyp group i.e. ∂_f(rel hyp group) is non-trivial for some f.

Open question: $G \curvearrowright S^2$ convergent group
Is it isomorphic to a subgroup of $PSL_2\mathbb{C}$?

Dynamical properties:

Classification of elements of a convergence group G.

$g\in G$

1. g is elliptic if $\overline{\{g^i\}_{i\in\mathbb{Z}}}$ is cmpt
2. g is parabolic if $\exists$ unique fixed point a for G and $\forall\, U,V$ open nbhds of a, there $\exists\, n\in\mathbb{Z}$ st $g^n(X\setminus U)\subseteq V$.
(If G discrete, then $g^n \to a$ uniformly on cmpt subsets in $X\setminus\{a\}$.)
3. g is loxodromic : there a 2 fixed pts a,b
$g^n(X\setminus\{b\}) \longrightarrow a$ on cmpt subsets of $X\setminus\{b\}$.
$X\setminus\{a,b\}/\langle g\rangle$ is cmpt.

Prop: If $\exists\, U$ open st ($\neq\emptyset, \neq Z$) $\bar{U}\subset g(U)$, then g is loxo with fixed points $a\in U$ and $b\notin \overline{gU}$.

Def^n: $G\curvearrowright X$ cvg group

$\Lambda_G := \{a\in X \mid \exists\,(a,b)\text{-collapsing seq}\}$

$\Omega_G := X\setminus \Lambda_G$

Prop: Λ_G is cmpt, G-inv
Ω_G is open, G-inv maximal open subset & $G \curvearrowright \Omega_G$ is proper.

Dynamical classification: (Caprace, Cornulier, Monod, Tessera).
$G \curvearrowright X$ cvgent

i) act^n is elementary i.e Λ_G = finite
— bounded i.e $\Lambda_G = \emptyset$ (G cmpt)
— parabolic i.e $\Lambda_G = \{*\}$.
— lineal i.e $\Lambda_G = \{*,*'\}$, $\exists g$ st $G/\langle g\rangle$ is cmpt.

2) non-elementary Λ_G is perfect cmpt space i.e no isolated pts
— focal : $\exists!$ fixed point $x \in X$ and loxodromic elt → e.g. $z \mapsto 2z+b$ on $\hat{\mathbb{C}}$, $b \neq 0$
— general : no fixed points
($\Rightarrow \exists$ free group)

Rmk : If G discrete, then there are no focal actions.
Gromov: $(G, |\cdot|)$ unbounded and acts by isom
$d(g_1 g_2) = \log(1 + |x_1 - x_2|)$
$\exists$ gives parabolic action because of this metric.

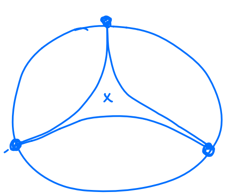
Peter Haïssinsky II.

- X metrizable cmpt set
- G discrete convergence group
  i.e $G < \mathrm{Homeo}(X)$ s.t.
  (1) $G \curvearrowright \Theta(X)$ is properly discontinuous
  (2) (g_n) is an ∞ seq, then $\exists$ subseq $(n_k) \to \infty$, $a, b \in X$ st
  $g_{n_k}|_{X \setminus \{a\}} \longrightarrow b$.

Thm. (Yaman, Gerasimov)
If the action on distinct pairs is cocmpt, then G is rel hyp.

Thm (B. Sun)
If G discrete convergence group, then G is acylindrically hyperbolic.

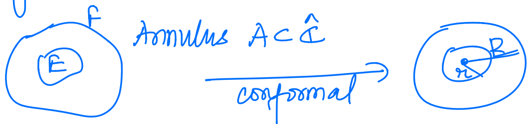
Thm (Bowditch) If the action of G on $\Theta(X)$ is cocmpt, then G is word hyp.
moreover, $\partial G \underset{\text{homeo}}{\cong} X$



a triple in X determines
a ~~too~~ bounded set in the "interior".
pd & cocmptness passes to "interior"

Digression: $\hat{\mathbb{C}}$

Annulus $A \subset \hat{\mathbb{C}}$ $\qquad \{r < |z| < R\}$



$$\text{mod}(A) = \tfrac{1}{2\pi} \log(R/r)$$

(a) for any $g \in \text{Möb}$, $\text{mod}(gA) = \text{mod}(A)$

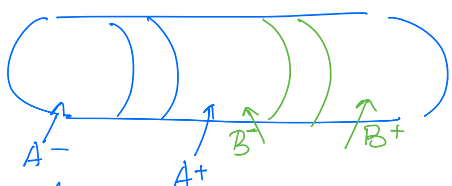
(b) If $\text{mod}(A) \to \infty$, then $\min\{\text{diam}\,E, \text{diam}\,F\} \to 0$.

<u>Def</u>^{n}: An annulus $A = (A^+, A^-)$ $\quad A^\pm$ closed subsets of X

$A^+ \cap A^- = \emptyset$

$X \setminus (A^+ \cup A^-) \neq \emptyset$

- $E \subset X$, say $E < A$ if $E \subset \text{int}(A^-)$; $E > A$ if $E \subset \text{int}(A^+)$
- two annuli are nested : $A < B$ if $A^+ \cup B^- = X$



let $\mathcal{A}$ be a collection of annuli in X

(E, F) an annulus

$$\text{mod}_{\mathcal{A}}(E,F) = \sup\{n \geq 0 \mid E < A_1 < A_2 \dots < A_n < F\}$$

Cover X by finitely many open sets $U_1, \dots U_k$

$$\mathcal{A}_0 = \{(\bar{U}_i, \bar{U}_j) \mid \bar{U}_i \cap \bar{U}_j = \emptyset,\ X \setminus (\bar{U}_i \cup \bar{U}_j) \neq \emptyset\}$$

$$\mathcal{A} = G \cdot \mathcal{A}_0$$

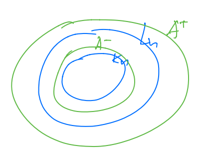
Bowditch's work.

$\therefore \text{mod}_{\mathcal{A}}$ is invariant under group action.

<u>Fact</u>: $(K_n, L_n)_{n \geq}$ annuli in X st $\forall n\ \exists A_n \in \mathcal{A}$ st $K_n < A_n < L_n$

if $\{A_n\}$ is ∞ then, upto a subseq,

then $\min(\text{diam}(K_n), \text{diam}(L_n)) \to 0$



<u>proof</u>: since finitely many orbits of annuli in $\mathcal{A}$

$\exists A \in \mathcal{A}$, $g_n \in G$ st $A_n = g_n(A)$

if $\{A_n\}$ ∞ then $\{g_n\}$ ∞ seq

$\therefore$ we get a collapsing seq i.e $\exists\, a, b$ st $g_n|_{X \setminus \{a\}} \to b$

if $a \in A^-$ then $g_n|_{A^+} \longrightarrow b$,

we have $g_n|_{A^+} \supset L_n$ $\quad \therefore L_n$ contained in small nbd of b.

$\therefore \operatorname{diam}(L_n) \to 0$.

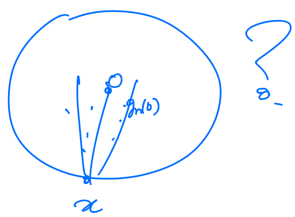
if $a \notin A^-$: then $g_n|_{A^-} \longrightarrow b$

so same holds for K_n.

<u>Corollary</u>: If $\operatorname{mod}_{\mathcal{A}}(E,F) = \infty$ then E or F is a single point.

<u>Converse?</u>

<u>Def</u>^{n}: $x \in X$ is a conical point if $\exists$ (x,b) collapsing seq (g_n) st $g_n(x) \not\longrightarrow b$!



<u>lemma</u>: a is conical, $g_n|_{X\setminus\{a\}} \longrightarrow b$ & $g_n(a) \longrightarrow c \neq b$

Assume $\exists A \in \mathcal{L}$, $b < A < C$

Then $\forall K \subset X\setminus\{a\}$, $\operatorname{mod}(\{a\}, K) = \infty$.

<u>Notation</u>: $x,y,z,w \in X$

$(x,y \mid z w) := \operatorname{mod}(\{x,y\}, \{z,w\})$

<u>Lemma</u>: $\exists\, k > 0$ st $\forall x,y,z,w$

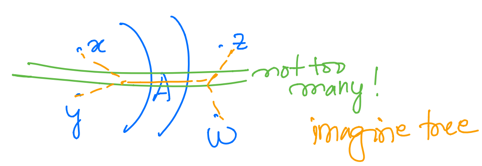
If $(xy \mid zw) > 1$, then $(xz \mid yw) > k$

<u>proof</u>: By contradict^n:

(x_n, y_n, z_n, w_n) st

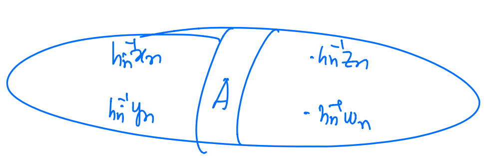
$(x_n y_n \mid z_n w_n) \geq 1$

But $(x_n z_n \mid y_n w_n) \geq n$.



$\mathcal{A}/G$ finite $\therefore \exists h_n \in G$, $A \in \mathcal{A}_0$ st. $\{x_n, y_n\} < h_n(A) < \{z_n, w_n\}$

$\{\underbrace{h_n^{-1}x_n}_{=x_n'}, \underbrace{h_n^{-1}y_n}_{=y_n'}\} < A < \{\underbrace{h_n^{-1}z_n}_{=z_n'}, \underbrace{h_n^{-1}w_n}_{=w_n'}\}$



$(x_n', z_n' \mid y_n', w_n') \longrightarrow \infty$

$\Rightarrow x_n', z_n' \longrightarrow u \in X$

cmpt sets $\hat{A}^+$ $\hat{A}^-$ $\Rightarrow u \in A^+ \cap A^- = \emptyset$

→ [now we will forget group act^n
just define ~~some~~ axioms using annuli to still get a hyp space (action?)]

Setting: X cmpt metrizable space
$\mathcal{A}$ a collection of annuli

(A1) (E,F) non-degenerate, $\mathrm{mod}(E,F) < \infty$
(at least two pts in E
" " " " in F)

(A2) $\exists\, k \geq 0$ st $(xy|zw) > k$ then all other cross-ratios are small.
$(xz|yw) \leq k$
$(xw|yz) \leq k$

(A3) $\forall x \in X$, $\forall K \subset X \setminus \{x\}$, $\mathrm{mod}(\{x\}, K) = \infty$

Thm (Bowditch) let $\rho : \Theta(X) \times \Theta(X) \longrightarrow \mathbb{R}_+$ defined as follows:
$\theta = (x_j)$, $\theta' = (x_j')$

$$\rho(\theta,\theta') = \max\{ (x_i x_j | x_\ell' x_k') \mid i,j,k,\ell \in \{1,2,3\} \}$$

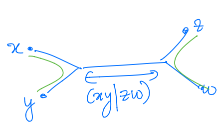
" distance b/w centers of tripods "

(1) If (A1) and (A2) hold, then $\exists\, C_m, C_g, \delta$ st $(\Theta(X),\rho)$ is an almost geodesic quasi-hyperbolic space.

- $\rho(\theta,\theta'') \leq \rho(\theta,\theta') + \rho(\theta',\theta'') + C_{metric}$
- $\forall\, \theta,\theta'$ $\exists\, \theta_0, \theta_1, \dots, \theta_n$ st $\theta_0 = \theta$, $\theta_n = \theta'$ and for any i,j
$$\left| \rho(\theta_i,\theta_j) - |i-j| \right| \leq C_{geodesic}$$
- set $(\theta_1,\theta_2)_\theta = \frac{1}{2}\left[\rho(\theta,\theta_1) + \rho(\theta,\theta_2) - \rho(\theta_1,\theta_2) \right]$
$\forall\, \theta_1,\theta_2,\theta_3,\theta_4$
$$(\theta_1,\theta_2)_{\theta_4} \geq \min\{ (\theta_1,\theta_3)_{\theta_4}, (\theta_1,\theta_2) \} - \delta$$

(2) If (A1), (A2), (A3) hold, then $\partial(\Theta(X),\rho) \simeq_{homeo} X$

Proposition: (Approximation by trees).
Assume
- (A1), (A2) hold
- $\forall\, n \geq 4$, $\exists\, C_n$ st $\forall$ finite set $F \subseteq X$ of cardinality n
$\varphi : F \longrightarrow T$ metric tree st



$\forall\, x,y,z,w \in F$, $\left| d_T([\varphi(x),\varphi(y)],[\varphi(z),\varphi(w)]) - (xy|zw) \right| \leq C_n$