YGGT - Anna Erschler (Talk 1)

Monday, February 17, 2020 4:21 PM

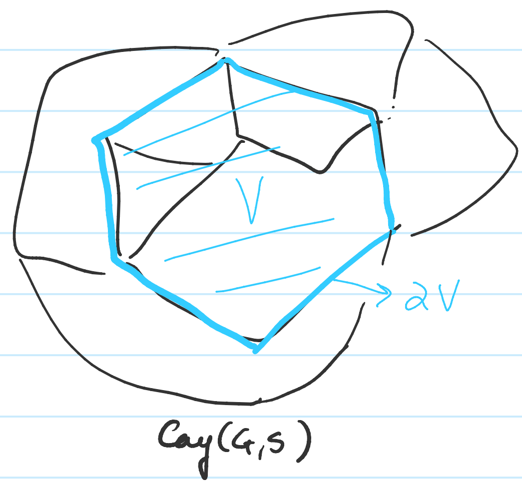
Growth and amenability of groups

— Anna Erschler.

G = fin. gen. group, S = generating set
$Cay(G,S)$ = Cayley graph

$V \subset G$ finite subset of G

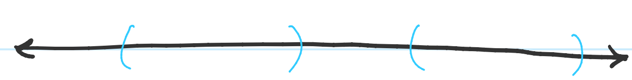
Can represent the subset V in the Cayley graph



∂V = boundary of V
$= \{ g \in G \mid d_{G,S}(g, V) = 1 \}$

Isoperimetric Inequality –
We know the size of bdry ∂V and we want to minimize the size of the set V.

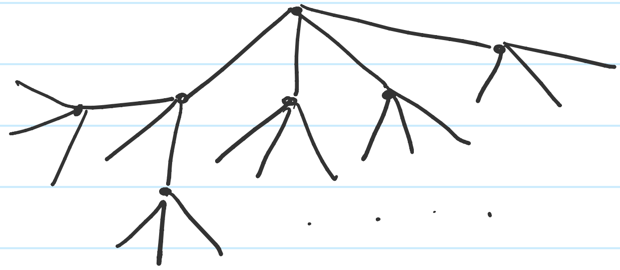
Ex 1: $G = \mathbb{Z}$ $S = \{\pm 1\}$



$V \subset Cay(G,S)$
∂V has at least 2 points, $|\partial V| \geq 2$
If $|\partial V| = 2$ then V is an interval.

Ex 2: $G = F_n$. free group on n generators

$Cay(F_n) =$

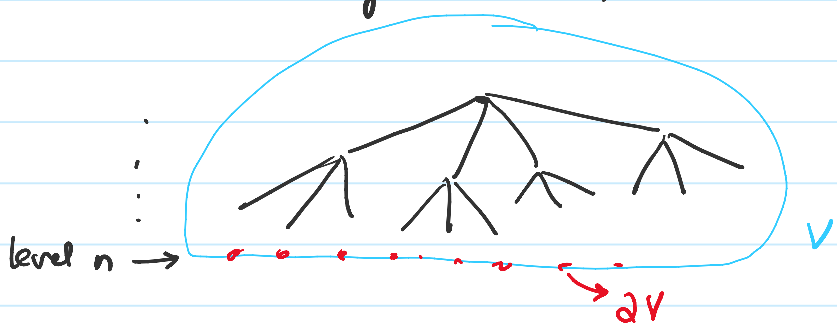


<u>Fact</u>: Free groups are non-amenable

<u>Def^n</u> Let G: fin. generated.

G is said to be <u>non-amenable</u> if $\exists$ constant C such that for any subset, $V \subset G$

$$\frac{\# \partial V}{\# V} \geq C$$



Let $B(e,n)$: ball of radius n.

$V_{G,S}(n) = \# B(e,n)$

<u>Observe</u>: $\partial V(n) = \#$ points at level $n = V(n) - V(n-1) = 2m(2m-1)^{n-1}$

$$V(n) = \sum_{i=0}^{n-1} 2m(2m-1)^i$$

Can't do better than ball.

Claim: Can't do better than a ball.

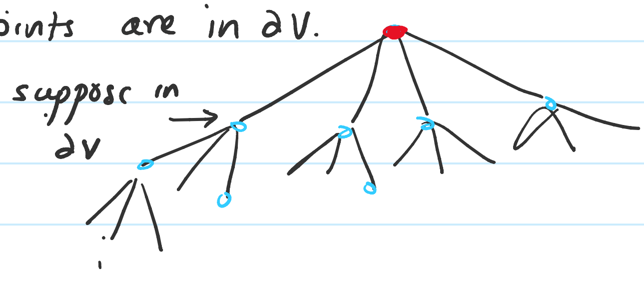
Let V be a subset with $\# V = V(n)$

<u>Claim</u>: $\# \partial V$ is at least as large as $\# B(e,n)$

Assume that not all points are in ∂V.

There is some pt (•) that all its neighbours in V

If a pt in ∂V, then it has missing neighbours. Pick a pt at a lower level and move it away



This process brings V closer to a ball $B(e,n)$.

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Ex 3 : Abelian group : $\mathbb{Z}^2$

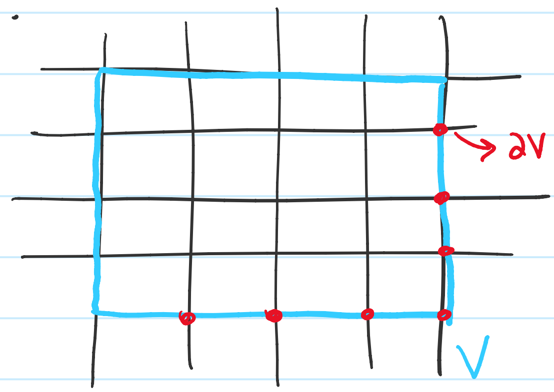
What are sets V with small ∂V ?

(a)

Let V be the square of side length n

$$\# V = (n+1)^2$$

$$\# \partial V = 4n$$



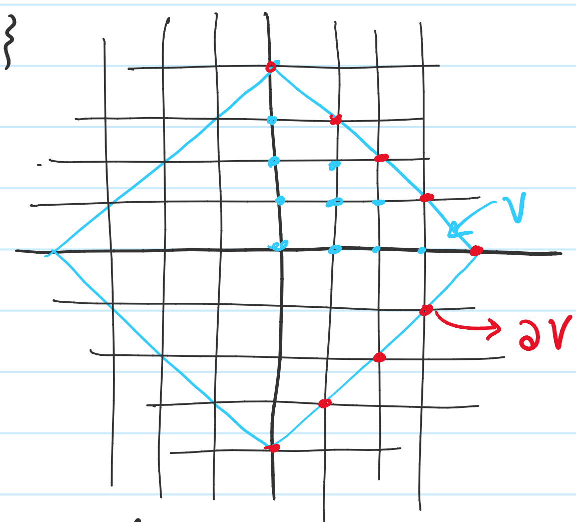
Fact : $\mathbb{Z}^2$ is amenable

Generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$

(b) Let $V = B(e,n)$

$$\# \partial V = 4n$$

$$\# V = (n+1)^2 + n^2$$



Observation : Balls are optimal

But this is not the case in general.

Lemma : Suppose $\dfrac{\# \partial B(e,n)}{\# B(e,n)} \longrightarrow 0$ as $n \to \infty$

Then then $V(n) = \# B(e,n)$ is subexponential

Proof : $V(n) = \dfrac{V(n)}{V(n-1)} \cdot \dfrac{V(n-1)}{V(n-2)} \cdot \ldots \cdot V(1)$

$$< (1+\varepsilon) \cdot (1+\varepsilon) \cdot \ldots$$

$$\leq (1+\varepsilon)^n \cdot C_\varepsilon$$

$\boxtimes$

Lemma 2: Suppose $V(n)$ is subexponential
$\exists\, n_i$ s.t. $\dfrac{\#\partial B(e,n_i)}{\# B(e,n_i)} \longrightarrow 0$

Open questions -

1) Assume $V(n)$ is subexponential.
   Is it true that $\dfrac{\#\partial B(e,n)}{\# B(e,n)} \longrightarrow 0$?

Remark: If G is virtually nilpotent, then
$$\underbrace{V(n) \sim c n^d}_{\text{(Pansu)}} + \underbrace{O(n^{d-1})}_{\text{Open!}}$$

* known for $O(n^r)$, $d-1<r<d$

2) G: exponential growth.
   Is it true that $\forall n$. $\dfrac{\#\partial B(e,n)}{\# B(e,n)} \geq$ constant ?

Recall:
Def^n: Følner sets - V such that $\dfrac{\#\partial V}{\# V}$ is small.

Fact: There are many amenable groups of exponential growth

Amenable $\Rightarrow$ $\exists$ sequence V_i such that $\dfrac{\#\partial V_i}{\# V_i} \to 0$

Four elementary operations
- taking subgroups
- taking quotients
- taking extensions
- $\bigcup_{G_i \subseteq G_{i+1}} G_i$

Definition –

Let G = countable, f.g. G is amenable iff all the finitely generated subgroups are amenable.

Note: Does not depend on generating set S.

* Amenability is invariant under the four operations

Example. If G is finite, G is amenable.

Def^n: G is elementary amenable if G is obtained from finite abelian group by the four operations.

Example: Solvable groups

Thm (Chou) G: f.g. infinite and elementary amenable. Then G be of polynomial growth or of exponential growth.

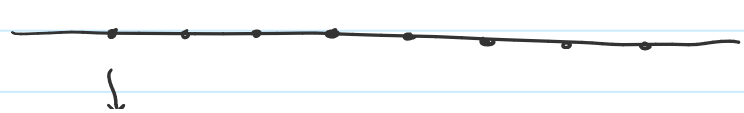
* Wreath product $A \wr B = A \times \sum_{A} B$

(A acts by shifts)

Example: $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \ltimes \sum_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ → called Lamplighter groups

has exponential growth

$(z, f) \in \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$



$z \in \mathbb{Z}$, $f: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$

$z \in \mathbb{Z}$, $f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$
finitely supported

lamps on (1) or off (0)

Observe: $V(2n) \geq 2^n$

$2n$ steps
obtains all configurations

$$\Omega_n = \{ (z,f) : -n \leq z \leq n, \ \operatorname{supp}(f) \subset [-n,n] \}$$

$$\# \Omega_n = 2^{2n+1} (2n+1)$$

$$\partial \Omega_n = \{ (z,f) \in \Omega_n : z = -n \text{ or } n \}$$

$$\# \partial \Omega_n = 2^{2n+1} \cdot 2$$

Def^n: $\mathrm{Fol}(n) = \min \left\{ \# V : \dfrac{\# \partial V}{\# V} \leq \dfrac{1}{n} \right\}$

Thm (Coulhon, Saloff-Coste) G: fin. gen.
$\exists\, C, B$ (depend on G, S) such that $\mathrm{Fol}(n) \geq V(Cn) + B$

Gap conjecture (Grigorchuk)

1) (Growth) G: f.g. group. Then - either $V(n) \leq C \cdot n^d$
- or $V(n) \geq \exp(n^a)$
for some a depending on

(Strong version) $a = \frac{1}{2} - \varepsilon$

2) (Folner functions) $\mathrm{Fol}(n)$ - either $\leq Cn^d$
- or $\geq \exp(Cn)$

Proposition G: virtually nilpotent, then $V(n) \le C n^d$

Thm (Gromov) If G has polynomial growth then G is

Question: Can the assumption in the above theorem be weakened?

Remark: Three proofs –

1) Assymptotic cones (original proof by Gromov)
2) Harmonic functions (Kleiner)
3) HFD (Ozawa)

Def^n: G is nilpotent if $\exists k$ such that $G_k = \{e\}$

$G \supset [G,G] \supset [G,[G,G]] \supset \cdots \supset G_k = \{e\}$ terminates

equivalently,
if $Z(G)$ = center,

Example:
Upper uni-triangular matrices $\begin{pmatrix} 1 & * & * & \boxed{*} \\ 0 & 1 & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ $\to Z(G)$

Ex.: $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ - step 1 nilpotent

$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ - step 2

We work in the group of matrices
Generators $s_1, \ldots, s_k$: $\begin{pmatrix} 1 & * & & 0 \\ & 1 & * & \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix}$

$s_1, \ldots, s_k$

$$[s_i, s_j] = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} \qquad C = [A, B]$$

Asymptotic cones -

Ex. $\mathbb{Z}$

In steps - $d(G,S)/n$ - distance becomes smaller

* proof 1 only really works for polynomial growth.