YGGT - Anna Erschler (Talk 2)

Monday, February 17, 2020 4:21 PM

Talk 2

Four elementary operations gives an amenable group.

<u>Exercise</u> : G : amenable, $H \subset G$, then H is amenable.

Amenable but not elementary amenable —

Elementary amenable gps : <u>Examples</u> $\mathbb{Z} \wr \mathbb{Z}$, $\mathbb{Z}^d \wr \mathbb{Z}$

$\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$

Wreath products give many examples of amenable gps.

* Solvable groups can have large $Føl(n)$.

In fact, easy to construct V such that $\# V \leq \exp(Cn^d)$ w/ $\frac{\# \partial V}{\# V} \leq \frac{1}{n}$

and cannot to better.

<u>Example</u> :

1) Baumslag-Solitar, $G_B = \langle a, b \mid aba^{-1} = b^2 \rangle$ is solvable and exponential growth.

<u>Exercise</u> – $V_{G_B}(n)$ is exponential

– $Føl_{G_B}(n) \leq \exp(Cn)$

2) Polycyclic gps $G = \{ M \in SL_2(\mathbb{Z})$ w/ eigenvalues λ_1, λ_2 and $\lambda_1 > 1$ $(\lambda_1 \lambda_2 = 1 \Rightarrow \lambda_2 < 1) \}$

$\lambda_1 > 1 \Rightarrow G$ has exponential growth.

<u>Example</u> : elementary amenable but not solvable —

Infinite uni-upper triangular matrices

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$$G = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \right\}$$

G is not finitely generated.
G is generated by elementary matrices $A_i = a_{i,i+1}$

matrices have finitely many nonzero entries over the diagonal.
$\Rightarrow$ G is union of nilpotent groups (finite upper triangular matrices)
$\Rightarrow$ G is amenable.

We want fin. generated. Consider HNN extension —
$A_i \longrightarrow A_{i+1}$ is an automorphism
$t A_i t^{-1} = A_{i+1}$

$\Rightarrow$ $G_1 = \langle A_1, t \rangle$ is fin. gen.,
elementary amenable, but <u>not solvable</u>.
contains nilpotent gps of arb. large degree

<u>Example</u> : $Sym(\mathbb{Z})$ - finite permutations on $\mathbb{Z}$
$G = Sym(\mathbb{Z}) *_t$, $t \curvearrowright \mathbb{Z}$ $\quad z \mapsto z+1$

<u>Rmk</u>: Elementary amenable are more complicated than solvable
But there are many hard questions for solvable and even nilpotent groups, for example quasi-isometry questions.

<u>Question</u> - Can one recognize the group from the Cayley graph?

<u>Example</u>: $\mathbb{Z}$ and $D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$ have same Cayley graph

<u>Note</u>: $\mathbb{Z} = \langle ab \rangle \underset{f.i}{\leq} D_\infty$

<u>Known</u>: If Cayley graph looks like $Cay(\mathbb{Z})$, then it is

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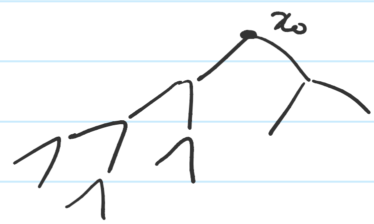
<u>Known</u>: for $\mathbb{Z}^2$

<u>Known</u>: If $Cay(G,S)$ has polynomial growth, then G is virtually nilpotent.

<u>Open Q</u>. Given G_1, G_2: nilpotent and quasi-isometric. Are G_1 and G_2 commensurable?

<u>Grigorchuk group</u>

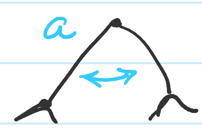
Rooted Binary tree:



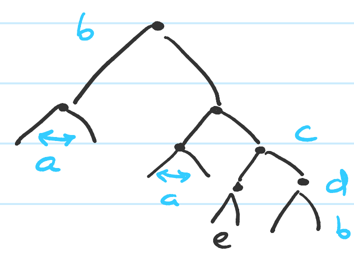
G acts on T by isometries with fixed pt x_0

$G = \langle a, b, c, d \rangle$:= first Grigorchuk group

a: permutation of two branches



b, c, d: defined recursively as follows:

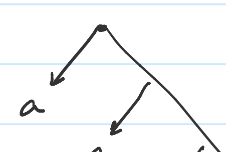


b: on first branch like a
on second branch like c

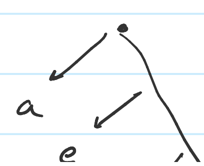
c: on first branch like a
on second branch like d

d: on first branch like e = trivial.
on second — " — b

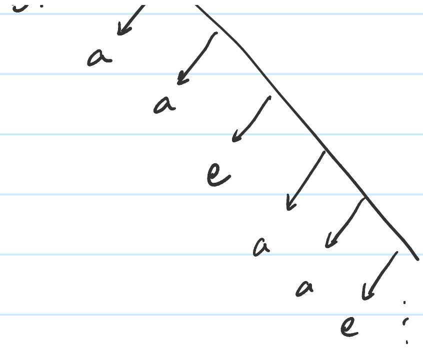
b:

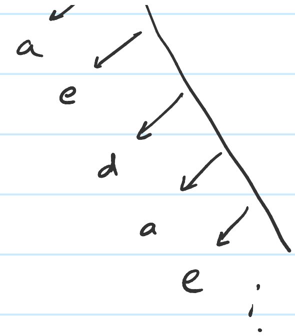


c:



d:





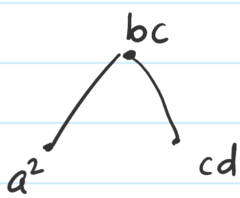
Growth (by defⁿ) is intermediate (i.e. not polynomial, nor exponential)

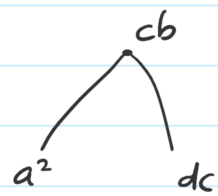
Lemma 0: In G_{Grig}

$a^2 = b^2 = c^2 = d^2 = e$

b, c, d commute $\Big\}$ b, c, d generate $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

can be proven simultaneously $\left\{ \begin{array}{l} a = bc \\ cd = dc \\ bd = db. \end{array} \right.$





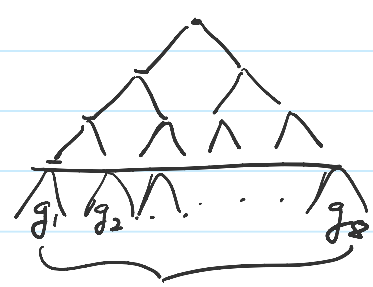
Main Lemma (Grigorchuk, Bartholdi) $V(n) \leq \exp(n^\alpha)$

$\alpha < 1$ $\alpha = 0.76\ldots$

Proof:

Consider the action at third level.

$H = \mathrm{Stab}(3) \to$ level 3



$2^3 = 8$ elements

$h \in H$ $h = (g_1, \ldots g_8)$

$h \in H \quad h = (g_1, \dots, g_8)$ $\qquad 2^3 = 8$ elements

$l_{G,S}$ = word metric with respect to $S = \langle a, b, c, d \rangle$

<u>Lemma 1</u>:

$$l_{G,S}(g_1) + l_{G,S}(g_2) + \cdots + l_{G,S}(g_8) \le \lambda\, l(h) + \text{constant} \quad (\lambda < 1)$$

Action is contracting

<u>Lemma 2</u>: G, H: subgroup of finite index in G

$H \to G^k$ is contracting, that is,

$h \mapsto (g_1, \dots, g_k)$ $\qquad \sum l_{G,S}(g_i) \le \lambda\, l_{G,S}(h) + \text{const}$, $\lambda < 1$

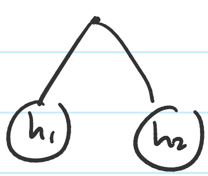
Then $\exists\, \alpha < 1$ (depends on λ) such that $V_{G,S}(n) \le \exp(c n^{\alpha})$

<u>Proof of Lemma 1</u>: $h \in H$, $l_{G,S}(h) = n$

Then $h = * a * a * \cdots a *$, where $* = b, c, d$

(by Lemma 0) $\qquad (bc = d)$

$h \in \mathrm{Stab}(1)$ $\qquad l(h_i) \le N_* = $ (number of $*$s)



$$\le \begin{cases} \frac{n}{2} + 1 & \\ \frac{n}{2} & \text{if } n \text{ divisible by } 2 \end{cases}$$

If $h \in \mathrm{Stab}(1)$, n is divisible by 2.

$$\Rightarrow \quad l(h_1) + l(h_2) \le n$$

Similarly, $l(g_1) + \cdots + l(g_8) \le n$

First case: Let N_b, N_c, N_d: number of b, c, d among $*$s

Suppose $N_a \geq \frac{1}{3}(N_b + N_c + N_d)$

$l_{G,S}(h_1) + l_{G,S}(h_2) \leq \left(1 - \frac{1}{6}\right) n$

$\frac{N_c}{N_b + N_c + N_d} \geq \frac{1}{3}$ then at next level $\frac{N_d}{N_b + N_c + N_d} \geq \frac{1}{6}$

Tomorrow: discuss lemma 2