YGGT - Anna Erschler (Talk 3)

Monday, February 17, 2020 4:21 PM

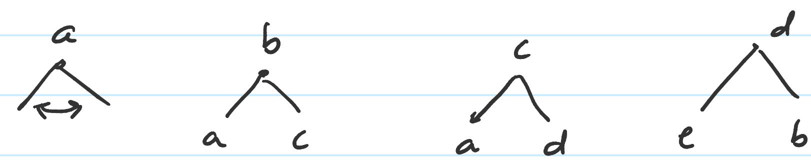
Talk 3 - Anna Erschler

First Grigorchuk group - G

G acts on a rooted tree

Generating set, $S = \{a, b, c, d\}$

action on tree:



$h \in \mathrm{Stab}(3)$

Goal: Find the best possible contraction coefficient upper bound is given by Bartholdi.

Bartholdi considered weights on the generators.

Let weights be $\omega(a) = A$

$\omega(b) = B$

$\omega(c) = C$

$\omega(d) = D$

Instead of word metric, we will use weighted lengths

Given $g = s_1 \cdots s_n$, $\quad s_i \in \{a, b, c, d\}$

$$\ell_\omega(g) := \sum_{i=1}^{n} \omega(s_i)$$

Asymptotically the weighted metric is quasi-isometric to the usual word metric.

Adv. Needed gen. to act trivially at some level.

Now, the contraction will be same on all levels.

It is enough to consider first level.

$h \in Stab(1)$

$h = (g_1, g_2)$

Consider $\ell(g_1) + \ell(g_2)$

Let $h = * \, a * a * \cdots * a$.

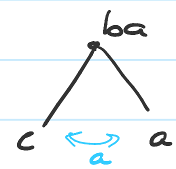
If $h \in Stab(1)$, then length(h) is even.

So, $h = (*a)(*a) \cdots (*a)$

It is enough to analyze $h = *a$, where $* = b, c, d$.

If $h = ba$, then $\ell_w(h) = B + A$

On the branches of h :



$\leftarrow \ell_w = B+A$

$\leftarrow \ell_w = C+A$

Want a λ, A, B, C so that following inequalities hold:

- $h = ba$ $\quad A + C \leq \lambda(B+A)$
- $h = ca$ $\quad D + A \leq \lambda(C+A)$
- $h = da$ $\quad B \leq \lambda(D+A)$

Goal: Find minimal possible λ.

Suppose $\exists \lambda$ that satisfies the above inequalities. Fix such λ.
We want to find A, B, C, D

Constraint : $A, B, C, D > 0$

We wish to solve the inequalities, but by allowing λ to change a, it is equivalent to solving the following system of equations

(1) $A + C = \lambda(B+A)$

(2) $D + A = \lambda(C+A)$

$$(1) \quad A+C = \lambda(B+A)$$
$$(2) \quad D+A = \lambda(C+A)$$
$$(3) \quad B = \lambda(D+A)$$

(1) & (3) $\to$ $A+C = \lambda(A+\lambda(D+A)) = A(\lambda+\lambda^2) + D\lambda^2 \qquad -(4)$

(2) & (4) $\to$ $D+A = \lambda(\ A(\lambda+\lambda^2) + D\lambda^2\ )$

$$= A(\lambda^2+\lambda^3) + D\lambda^3$$

$$D(1-\lambda^3) = A(\lambda^3+\lambda^2-1)$$

Put $A := 1-\lambda^3$, $D := \lambda^3+\lambda^2-1$, $B := \lambda^3$, $C := \lambda^3+\lambda-1$

$$\left[\begin{array}{c} B+A = 1 \\ A+C = \lambda \Rightarrow C = \lambda - A \end{array}\right]$$

We cannot have $\lambda = 1$.

What about $\lambda = 0.8$? Observe that $\lambda = 0.8 \Rightarrow A, B, C, D > 0$

Yet $\lambda = 0.8$ does not work.

Reason: With the weighted metric, it is not necessarily true that $h = *a * a \ldots *a$

To ensure that, we need additional conditions.

$$B \le C+D, \quad C \le B+D, \quad D \le B+C.$$

$$\Rightarrow \lambda^3+\lambda^2+\lambda-2 \ge 0$$

So $\lambda = 0.8$ does not work!

Minimal λ : solve $-$ $\lambda^3+\lambda^2+\lambda = 2 \rightsquigarrow$ has 2 cplx roots & 1 positive real root

Let λ_0 = positive root of this eqn

$\sim 0.8105\ldots$

Lemma 2:

$H \subset G$, finite index

$H \xrightarrow[\text{injective}]{} G^d \qquad \exists \lambda \qquad \ell(h) \le \lambda \sum \ell(g_i) + \text{constant}$

$(\lambda < 1)$

injective

$(\lambda < 1)$

$$\alpha = \frac{\ln d}{\ln d - \ln \lambda}$$

Growth of $G \leq \exp(cn^{\alpha})$

$V(r) = \# B(e,r)$

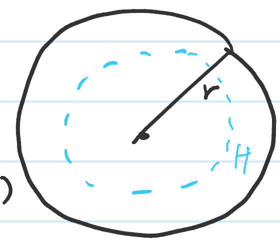
$H \cap B(e,r)$

Growth $\leq$ const $\cdot \# H \cap B(e,r)$

Want to count $\# \{ h \in H \cap B(e,r) \}$

$h = (g_1, \ldots, g_d)$

B(e,r)



<u>Corollary</u>: For first Grigorchuk group, $\alpha_0 = \dfrac{\ln 2}{\ln 2 - \ln \lambda_0} = 0.767\ldots$

<u>Open questions</u> – Can we have a finitely presented group of intermediate growth?
(Milnor)

<u>Rmk</u>: First Grigorchuk group is residually finite.

Central extensions of Grigorchuk gps are not res. finite.

(Nekrashevych) $\exists$ simple groups of intermediate growth.

$*$ Action on tree

$G \curvearrowright$ rooted tree $\rightsquigarrow$ consider action on the boundary of tree

= Cantor set

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* Generalizations of Grigorchuk growth estimate —

- (E., Tianyi, Zhang) For 1st Grig. gp, Growth $\geq \exp(c n^{\alpha_0 - \varepsilon})$ $\forall \varepsilon$

- $f(n) \geq n^{\alpha_0}$ non decreasing & subadditive
$\exists G$: $V(n) \sim \exp(f(n))$ (Erschler, Bartholdi)

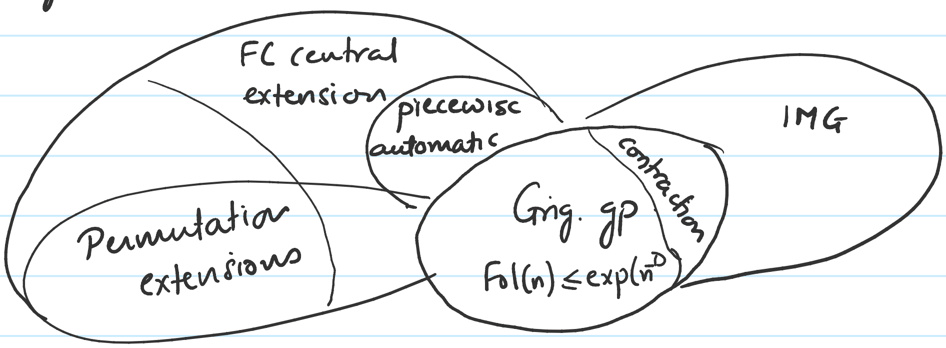
w/o any extra cond^n (T. Zhang 2020)

<u>Rmk</u> : $V(n+m) \leq V(n) V(m)$
$\ln V(n+m) \leq \ln V(n) + \ln V(m)$
If $\lim \frac{\ln V(n)}{n} > 0$, we say the growth is exp.

* Not all groups of int. growth admit contraction.
$\exists$ ctbly many groups that have contraction.

$\exists$ continuum of groups G_ω action on trees. They don't have contraction in the standard sense, but have contraction in a more general sense — not all words have contraction

Ways to extend the Grigorchuk group —



Wreath product: $A \wr B = A \ltimes \sum_A B$

Permutational wreath product $\Big\{$ A acts on X $\quad A \ltimes \sum_X B = (A, X) \wr B$

If A, B amenable, then $(A, X) \wr B$ amenable

$G \curvearrowright$ tree $\quad$ eg. $G =$ 1st Grig. group

$\partial T = \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ $\qquad X =$ some orbit

$(G_{Grig}, X) \wr \mathbb{Z}/2\mathbb{Z}$

$(G_{Grig}, X) \wr B$ (B: finite) $\Rightarrow$ growth in $\exp(n^{\alpha_0})$

[different actions on different parts of the tree to get different growths]

Groups with FC center $\rightarrow$ finitely many conjugates.

<u>Lemma</u> (T. Zhan) G has subexp. growth, $H =$ FC center extension

Then H has subexp growth

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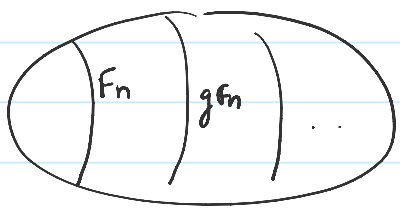
Then H has subexp growth.

Exp. growth	Subexp growth
Non-amenable	Amenability
$F_n < G \Rightarrow G$ - non-amenable	



$\Omega \subseteq G \qquad \Omega_i = \Omega \cap g_i F_n$

$\Omega = \bigcup \Omega_i$

$\# \Omega > c \, \# \Omega_i$

Question - G: exponential growth $\overset{?}{\Rightarrow}$ G contains Lipschitz embedding of a binary tree

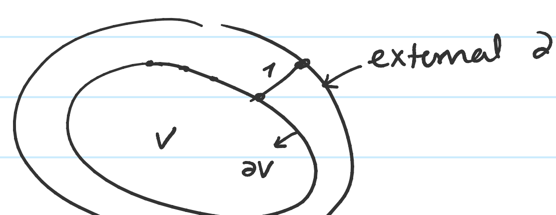
: (Rosenblatt)

Lemma: Inside every non-amenable group, there is a tree.

Pf G: non-amenable $\quad \# \partial \Omega \geq c \, \# \Omega$

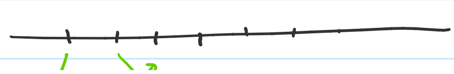
Changing gen. set, we can assume $c \geq 2$.

External boundary = pts at dist = 1 from the ∂



(Marriage lemma)

Bipartite graph



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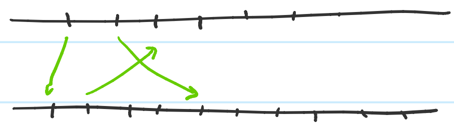
Bipartite graph



← build a bipartite graph

By extension of marriage lemma,

$\forall$ point on ∂V, choose 2 pts on external ∂

build a tree

Question: Where to find a group with exponential growth without a tree?